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LETTER TO THE EDITOR

Acoustic rectification and the virial theorem

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Abstract. Rectification of acoustic waves corresponding to each mode of propagation in crystalline solids (modal acoustic radiation-induced static strains) is predicted directly from the virial theorem for an elastic continuum.

The notion of acoustic rectification (radiation-induced static strains) is strongly associated with that of acoustic radiation stress. Both acoustic rectification and radiation stress have been the subjects of considerable controversy for a large part of the present century. Lord Rayleigh [1] introduced the concept of acoustic radiation stress in analogy to electromagnetic radiation stress, but his derivation was challenged by Brillouin [2]. Although Brillouin [2] predicts the existence of an acoustic radiation stress in solids and 'laterally confined' fluids, his theory leads to a prediction of a null acoustic rectification. Gol'dberg [3] argues that the radiation stress in 'laterally unconfined' fluids is zero and, inferentially, so is the acoustic rectification. Both Thurston and Shapiro [4] and Thompson and Tiersten [5] predict the existence of a non-zero acoustic rectification but they predict different amplitudes of the rectified signal. Chu and Apfel [6] identify an 'acoustic straining' associated with the radiation pressure in laterally confined fluids. Cantrell and Yost [7] show that the radiation stress generated by an acoustic wave propagating in a crystalline solid has an accompanying radiation-induced static strain for each independent propagation mode of the crystal. The purpose of this paper is to show that such static strains or rectifications are predicted from the virial theorem for an elastic continuum. The virial theorem is statistical in nature and when extended to include conservative nonlinear elastic media leads to an expression directly involving energy-dependent quantities which define the static strain from first principles. It is thus more forgiving of ill-defined or speculative boundary conditions that are responsible in part for the confusion surrounding the subject.

We begin by defining a quantity (we assume Einstein summation convention of repeated indices)

$$G = \int_{V_0} \pi_i u_i \, \mathrm{d}V_0 \tag{1}$$

where π_i is the i-component of the momentum density of a sinusoidal acoustic wave, u_i is the corresponding component of the particle displacement vector, and V_0 is a finite volume fixed in Lagrangian space in which π_i and u_i are continuous for all time. The total time-derivative of (1) is

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \int_{V_0} (\pi_i \dot{u}_i + \dot{\pi}_i u_i) \, \mathrm{d}V_0. \tag{2}$$

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The term $\pi_i \dot{u}_i$ in (2) is equal to twice the kinetic energy density T of the acoustic wave. Using the relations (equations of motion)

$$\dot{\pi}_i = \frac{\partial \sigma_{ij}}{\partial a_j} \tag{3}$$

where σ_{ij} are the components of the Boussinesq (first Poila–Kirchhoff) stress tensor and a_j are the Lagrangian coordinates, we recast (2) in the form

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \int_{V_0} \left(2T + \frac{\partial \sigma_{ij}}{\partial a_j} u_i \right) \mathrm{d}V_0. \tag{4}$$

We define the time-average of a quantity $Q(a_k, t)$ by

$$\langle Q \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t Q(a_k, t')\, \mathrm{d}t' \tag{5}$$

and designate the time-averaging operation by the angular brackets $\langle\ \rangle$. We assume here that the quantity $Q(a_k, t)$ is non-zero for all time and that the limit is taken at a given fixed spatial position a_k. Time-averaging (4), we find that the term $\langle \mathrm{d}G/\mathrm{d}t \rangle$ vanishes for any value of V_0 as the result of the boundedness of G. Using the commutativity of the time-averaging and spatial integration operations, we obtain the virial theorem for an elastic continuum in the form

$$\langle T \rangle = -\frac{1}{2} \left\langle \frac{\partial \sigma_{ij}}{\partial a_j} u_i \right\rangle \tag{6}$$

where the right-hand side of (6) may be appropriately called the *virial of Clausius for an elastic continuum.*

The virial theorem may be written in a form more expedient for present purposes by using the identity. $(\partial \sigma_{ij}/\partial a_j) u_i = [\partial(\sigma_{ij} u_i)/\partial a_j] - \sigma_{ij}(\partial u_i/\partial a_j)$, together with Green's theorem in (4). Time-averaging the resulting expression and using the distributive properties of the time-averaging operation we obtain

$$0 = \int_{V_0} \left\langle 2T + \sigma_{ij} \frac{\partial u_i}{\partial a_j} \right\rangle \mathrm{d}V_0 + \int_{S_0} \langle \sigma_{ij} u_i \rangle\, \mathrm{d}S_j \tag{7}$$

where S_0 is a closed surface bounding the volume V_0. We consider finite-amplitude acoustic plane waves propagating in a lossless medium the spatial dimensions of which are sufficiently large that no reflections occur and we assume that the volume contains no acoustic sources. In such a lossless propagation medium the Hamiltonian is conservative and under the conditions delineated it is reasonable to assume that the time-averaged kinetic energy density $\langle T \rangle$ and the term $\langle \sigma_{ij}(\partial u_i/\partial a_j) \rangle$ (time-averaged product of stress and strain related to time-averaged potential energy density) are constant and uniform throughout the region of interest. If we make the reasonable assumption that $\langle \sigma_{ij} u_i \rangle$ is also constant in the region of interest, the surface integral of (7) vanishes for all closed surfaces S_0. We thus obtain from (7)

$$\langle T \rangle = -\frac{1}{2} \left\langle \sigma_{ij} \frac{\partial u_i}{\partial a_j} \right\rangle. \tag{8}$$

We now expand the Boussinesq stress σ_{ij} in (8) in terms of the displacement gradients $(\partial u_i/\partial a_j) = u_{ij}$ as [8]

$$\sigma_{ij} = A_{ijkl}u_{kl} + \tfrac{1}{2}A_{ijklmn}u_{kl}u_{mn} + \cdots \tag{9}$$

where A_{ijkl} and A_{ijklmn} are the Huang coefficients. Transforming coordinates in (9) such that the transformed Lagrangian coordinate a_i is always along the direction of wave propagation we obtain the expression [7] (no sum on ϵ in all following equations)

$$\tau_{\epsilon 1} = \mu_\epsilon \left(\frac{\partial P_\epsilon}{\partial a_1}\right) - \frac{1}{2}\mu_\epsilon\beta_\epsilon\left(\frac{\partial P_\epsilon}{\partial a_1}\right)^2 + \cdots \tag{10}$$

where $\tau_{\epsilon 1}$ is the transformed Boussinesq stress tensor, $(\partial P_\epsilon/\partial a_1)$ is the transformed displacement gradient, $\epsilon = (\alpha, N)$ is a mode index representing a wave of polarization $\alpha = 1, 2, 3$ and direction of propagation N. The constant $\mu_\epsilon = A_{ijkl}N_jN_lU_iU_k$ where the N_i are the Cartesian components of N referred to the original coordinate system and the U_i are the Cartesian components of the wave polarization direction referred to the same system. The constant $\beta_\epsilon = -(A_{ijklmn}N_jN_lN_nU_iU_kU_m/A_{ijkl}N_jN_lU_iU_k)$ is designated the modal acoustic nonlinearity parameter from its appearance in the nonlinear equations of elastic wave motion and serves as a quantitative measure of acoustic nonlinearity [9]. Performing a similar transformation on (8) we obtain the virial theorem in the transformed frame as

$$\langle K\rangle = -\frac{1}{2}\left\langle \tau_{\epsilon 1}\frac{\partial P_\epsilon}{\partial a_1}\right\rangle \tag{11}$$

where the transformed kinetic energy density $K = (1/2)\rho_0(\partial P_\epsilon/\partial t)^2$ and ρ_0 is the mass density of the unperturbed solid.

Substituting (10) into (11) and factoring the resulting expression we obtain the time-averaged expression

$$\left\langle\left\{\frac{1}{c_\epsilon}\frac{\partial P_\epsilon}{\partial t} + \frac{\partial P_\epsilon}{\partial a_1}\left[1 - \frac{1}{2}\beta_\epsilon\left(\frac{\partial P_\epsilon}{\partial a_1}\right)\right]^{1/2}\right\}\left\{\frac{1}{c_\epsilon}\frac{\partial P_\epsilon}{\partial t} - \frac{\partial P_\epsilon}{\partial a_1}\left[1 - \frac{1}{2}\beta_\epsilon\left(\frac{\partial P_\epsilon}{\partial a_1}\right)\right]^{1/2}\right\}\right\rangle = 0 \tag{12}$$

where the sound speed $c_\epsilon = (\mu_\epsilon/\rho_0)^{1/2}$ in (12). It is instructive to note that for the case where the nonlinearity parameter $\beta_\epsilon = 0$, setting each braced set of terms in (12) independently to zero yields the relationship between the particle velocity $(\partial P_\epsilon/\partial t)$ and the displacement gradient $(\partial P_\epsilon/\partial a_1)$ expected for linear waves propagating correspondingly in opposite directions. We expect then that for the case where β_ϵ is non-zero, setting the braced terms independently to zero yields the corresponding relationships for oppositely propagating nonlinear waves. We thus consider only the braced set of terms on the left in (12) for wave propagation along the positive a_1 axis and expand the terms under the square root in a power series to first order in the nonlinearity. We write

$$\left\langle\frac{1}{4}\beta_\epsilon\left(\frac{\partial P_\epsilon}{\partial a_1}\right)^2 - \left(\frac{\partial P_\epsilon}{\partial a_1}\right) - \frac{1}{c_\epsilon}\left(\frac{\partial P_\epsilon}{\partial t}\right)\right\rangle = 0. \tag{13}$$

To the same approximation used in obtaining (13) we may interchange the roles of $(\partial/\partial t)$ and $(\partial/\partial a_1)$ in (13) by using the operator relationship

$$\frac{\partial}{\partial t} \cong c_\epsilon \frac{\partial}{\partial a_1}. \tag{14}$$

Substituting (14) into (13) and using the distributive properties of the time-averaging operation, we get the expression

$$\left\langle \frac{\partial P_\epsilon}{\partial a_1} \right\rangle = -\frac{1}{c_\epsilon}\left\langle \frac{\partial P_\epsilon}{\partial t} \right\rangle + \frac{\beta_\epsilon}{4c_\epsilon^2}\left\langle \left(\frac{\partial P_\epsilon}{\partial t}\right)^2 \right\rangle. \tag{15}$$

The term $(\partial P_\epsilon/\partial t)$ in (15) vanishes as the result of the boundedness of the displacement P_ϵ. For the conservative, nonlinear system considered here we may write the time-averaged total energy density $\langle E^\epsilon \rangle = (\mu_\epsilon/c_\epsilon^2)(\partial P_\epsilon/\partial t)^2\rangle$. Substituting this expression into (15), we finally obtain

$$\left\langle \frac{\partial P_\epsilon}{\partial a_1} \right\rangle = \frac{\beta_\epsilon}{4\mu_\epsilon}\langle E^\epsilon \rangle. \tag{16}$$

The time-averaged displacement gradient $\langle \partial P_\epsilon/\partial a_1 \rangle$ is the modal acoustic radiation-induced static strain in the solid and is the quantitative measure of acoustic rectification.

It is apparent from the derivation that the acoustic rectification arises as a consequence of the cubic (anharmonic) term in the potential energy expansion with respect to the displacement gradients (quadratic term in (9) for the expansion of stress with respect to the displacement gradients). The direct dependence of the static strain on the nonlinearity parameter β_ϵ in (16) means that acoustic rectification is a nonlinear phenomenon. Equation (16) is derived without specific regard to boundary or initial conditions since only energy dependent terms pertaining to effectively boundless propagation media are involved in the derivation. The equation is found to be in agreement with experimental data obtained for acoustic waves propagating along the pure longitudinal mode propagation directions of single crystal silicon [10] and for propagation in vitreous silica and single crystal germanium [7]. An examination of previous theoretical research indicates that part of the controversy over the static strains can be traced either to neglecting terms corresponding to the second term in (15) relating the time-averaged displacement gradient to the time-averaged particle velocity or to imposing inappropriate boundary and initial conditions when solving for the static displacement directly from the nonlinear wave equation.

Finally, the experimental confirmation of modal radiation-induced static strains together with the statistical nature of the virial theorem suggest a connection between acoustic rectification and certain thermodynamic properties of crystals expressed in terms of stochastic nonlinear acoustic fields. This connection is the subject of a paper now in preparation [11].

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References

[1] Rayleigh, Lord 1902 *Phil. Mag.* **3** 338; 1910 *Phil. Mag.* **10** 364
[2] Brillouin L 1925 *Ann. Phys., Paris* **4** 528; 1925 *J. Phys. Radium* **6** 337
[3] Gol'dberg Z A 1971 *High Intensity Ultrasonic Fields* ed L D Rozenburg (New York: Plenum)
[4] Thurston R N and Shapiro M J 1967 *J. Acoust. Soc. Am.* **41** 1112
[5] Thompson R B and Tiersten H F 1977 *J. Acoust. Soc. Am.* **62** 33
[6] Chu B-T and Apfel R E 1982 *J. Acoust. Soc. Am.* **72** 1673
[7] Cantrell J H 1984 *Phys. Rev.* B **30** 3214
Yost W T and Cantrell J H 1984 *Phys. Rev.* B **30** 3221
[8] Huang K 1950 *Proc. R. Soc.* A **203** 178
[9] Cantrell J H Crystalline structure dependence of acoustic nonlinearity parameters *Phys. Rev.* B at press
[10] Cantrell J H, Yost W T and Li P 1987 *Phys. Rev.* B **35** 7780
[11] Cantrell J H, Thermal expansivity and stochastic nonlinear acoustic fields, in preparation